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Betti numbers of 3-dimensional Gorenstein Stanley-Reisner rings

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序

有限集合 $V = \{x_1, x_2, \ldots, x_v\}$ に対して、頂点集合 V 上の単体的複体 (simplicial complex) Δ を次の条件 (1)、(2) を満たす $\mathbf{2}^V$ の部分集合とする。但し、$\mathbf{2}^V$ は V の部分集合全体からなる集合とする。
(1) $1 \leq i \leq v$ に対して、$\{x_i\} \in \Delta$。
(2) $\sigma \in \Delta, \tau \subset \sigma \Rightarrow \tau \in \Delta$。

$\sharp(\sigma)$ で 有限集合 σ の濃度を表すことにする。Δ の元 σを Δ の面 (face) という。特に、$\sharp(\sigma) = i+1$ のとき、i-face という。$d = \max\{\sharp(\sigma) \mid \sigma \in \Delta\}$ とおき、Δ の次元 (dimension) を $\dim \Delta = d-1$ で定義する。

$A = k[x_1, x_2, \ldots, x_v]$ を体 上の v変数多項式環とする。$V = \{x_1, x_2, \ldots, x_v\}$ 上の単体的複体 Δに対して A のイデアル I_Δ を次のように定義する。

$$I_\Delta = (x_{i_1} x_{i_2} \cdots x_{i_r} | 1 \leq i_1 < i_2 < \cdots < i_r \leq v,\ \{x_{i_1}, x_{i_2}, \cdots, x_{i_r}\} \notin \Delta)$$

$k[\Delta] := A/I_\Delta$ をΔ の Stanley–Reisner 環という。

以後、A を $\deg x_i = 1$ として 次数付き環 $A = \bigoplus_{n \geq 0} A_n$ とみなす。すると、$k[\Delta]$ もまた、自然に A 上の次数付き加群 $k[\Delta] = \bigoplus_{n \geq 0} (k[\Delta])_n$ とみなせる。

$k[\Delta]$ の A 上の次数付き極小自由分解を

$$0 \longrightarrow \bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{h,j}} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{1,j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} k[\Delta] \longrightarrow 0$$

とする。ここで、h を $k[\Delta]$ の ホモロジー次元 (homological dimension) といい、$h = \mathrm{hd}_A(k[\Delta])$ とあらわす。このとき $v - d \leq h \leq v$が 成り立つことが知られている。各 $\beta_{i,j}$ を $k[\Delta]$ の (i,j) ベッチ数 ((i,j)-th Betti number)

といい、また、$\beta_i := \sum_{j\in\mathbf{Z}} \beta_{i,j}$ を $k[\Delta]$ の第 i ベッチ数 (i-th Betti number) という。Δ が $d-1$ 次元の Gorenstein complex ならば、$h=v-d$, $\beta_h = 1$ となる。

Upper Bound Theorem (§1 を見よ) からも想像されるように、cyclic polytope の boundary complex は Gorenstein complex の中である種の extremal な性質をもつことが期待される。そこで、作業仮説として次のような問題を考えてみた。

問題　単体的複体 Δ を v 個の頂点をもつ $(d-1)$ 次元の Gorenstein complex とする。そのとき

$$\beta_i(k[\Delta]) \leq \beta_i(k[\Delta(C(v,d))]).$$

が成立するか？ 特に、$d=3$ のとき、

$$\beta_i(k[\Delta]) \leq \frac{(v-1)(v-i-3)}{i+1}\binom{v-3}{i-1}, \ 1 \leq i \leq v-4,$$

が成り立つか？

筆者の知る限りにおいては、この問題の反例はまだ知られていないと思う。(ご存知の方は、是非御教示下さい。) そこで、本稿においては、この問題の $d=3$ の場合についての部分的解答を試みる。

以下、本稿の構成を述べる。まず§1 では、Stanley–Reisner 環の Betti 数を単体的複体のホモロジー群の言葉であらわす Hochster の公式について説明する。この公式が、Stanley–Reisner 環の Betti 数を組合せ論的手法を用いて研究できる基礎を与えている。続く§2 では、cyclic polytope の定義を与え、日比孝之氏 (大阪大学) との共同研究である cyclic polytope に付随する Stanley-Reisner 環の Betti 数を具体的にあらわす公式を与える。そして、最後の§3 では上記の問題についてにひとつの結果とその証明の概略を述べる。

§1. Hochster's formula

Given a subset W of V, the *restriction* of Δ to W is the subcomplex

$$\Delta_W = \{\sigma \in \Delta \mid \sigma \subset W\}$$

of Δ. In particular, $\Delta_V = \Delta$ and $\Delta_\emptyset = \{\emptyset\}$.

Let $\tilde{H}_i(\Delta; k)$ denote the i-th *reduced simplicial homology group* of Δ with the coefficient field k. Note that $\tilde{H}_{-1}(\Delta; k) = 0$ if $\Delta \neq \{\emptyset\}$ and

$$\tilde{H}_i(\{\emptyset\}; k) = \begin{cases} 0 & (i \geq 0) \\ k & (i = -1). \end{cases}$$

Hochster's formula [Hoc, Theorem 5.1] is that

$$\beta_{i_j} = \sum_{W \subset V,\ \sharp(W)=j} \dim_k \tilde{H}_{j-i-1}(\Delta_W; k).$$

Thus, in particular,

$$\beta_i^A(k[\Delta]) = \sum_{W \subset V} \dim_k \tilde{H}_{\sharp(W)-i-1}(\Delta_W; k).$$

Some combinatorial and algebraic applications of Hochster's formula have been studied. See, e.g., [B–H$_1$], [B–H$_2$], [H$_2$], [H$_3$], [H$_4$], and [T–H$_1$].

§2. Betti numbers of Stanley–Reisner rings associated with cyclic polytopes

In this section we briefly summarize the results in [T–H$_2$]. See [T–H$_2$] for the detailed information. See also e.g., [Brø] for general properties of cyclic polytopes.

Let $\mathbf{R}$ denote the set of real numbers. For any subset M of the d-dimensional Euclidean space $\mathbf{R}^d$, there is a smallest convex set containing M. We call this convex set the *convex hull* of M and denote it by conv$M$. For $d \geq 2$ the *moment curve* in $\mathbf{R}^d$ is the curve parametrized by

$$t \longmapsto x(t) := (t, t^2, \cdots, t^d), \qquad t \in \mathbf{R}.$$

By a *cyclic polytope* $C(v, d)$, where $v \geq d+1$ and $d \geq 2$, we mean a polytope $\mathcal{P}$ of the form $\mathcal{P} = \text{conv}\{x(t_1), \cdots, x(t_v)\}$, where $t_1, \cdots, t_v$ are distinct real numbers. It is well known that $C(v, d)$ is a simplicial d-polytope with the vertex set $\{x(t_1), \cdots, x(t_v)\}$, and its face lattice is independent of the particular values of t. Therefore its boundary complex is a simplicial complex and has the same combinatorial structure for any choices of vertices. We denote it by $\Delta(C(v, d))$.

The next theorem explains why the cyclic polytope is important.

UPPER BOUND THEOREM (cf. e.g., [Brø]) *Let $\mathcal{P}$ be a d-dimensional polytope with v vetices. Let C be a d-dimensional cyclic polytope with v vertices. Then we have*

$$f_i(\mathcal{P}) \leq f_i(C),$$

where $f_i(\mathcal{P})$ stands for the number of i-facces of a polytope $\mathcal{P}$.

We now compute the Betti numbers of a minimal free resolution of the Stanley-Reisner ring $k[\Delta(C(v,d))]$ of the boundary complex $\Delta(C(v,d))$ of the cyclic polytope $C(v,d)$.

We fix a field k .

If the dimension d is even, a minimal free resolution of $k[\Delta]$ is pure and the Betti numbers can be computed from the Hilbert function of $k[\Delta]$.

PROPOSITION 2.1 ([Sch]). *Let Δ be the boundary complex $\Delta(C(v,d))$ of the cyclic polytope $C(v,d)$, where $d \geq 2$ is even. Then a minimal free resolution of $k[\Delta]$ over A is of the form:*

$$\begin{aligned} 0 \longrightarrow A(-v) \longrightarrow A(-v+\frac{d}{2}+1)^{\beta_{v-d-1}} \longrightarrow \\ \cdots \longrightarrow A(-\frac{d}{2}-2)^{\beta_2} \longrightarrow A(-\frac{d}{2}-1)^{\beta_1} \longrightarrow A \longrightarrow k[\Delta] \longrightarrow 0, \end{aligned}$$

where for $1 \leq i \leq v-d-1$,

$$\beta_i = \binom{v-\frac{d}{2}-1}{\frac{d}{2}+i}\binom{\frac{d}{2}+i-1}{\frac{d}{2}} + \binom{v-\frac{d}{2}-1}{i-1}\binom{v-\frac{d}{2}-i-1}{\frac{d}{2}}.$$

Our formula on β_i in Proposition 2.1 is, in fact, a little bit different from the one in [Sch]. But it is easy to show that they are coincident.

Now we state the main theorem in this chapter.

THEOREM 2.2 ([T–H$_2$]). *Let Δ be the boundary complex $\Delta(C(v,d))$ of the cyclic polytope $C(v,d)$, where $d \geq 3$ is odd. Then a minimal free resolution of $k[\Delta]$ over A is of the form:*

$$0 \longrightarrow A(-v) \longrightarrow A\left(-v+\left[\frac{d}{2}\right]+2\right)^{b_{v-d-1}} \oplus A\left(-v+\left[\frac{d}{2}\right]+1\right)^{b_1} \longrightarrow$$

$$\cdots \longrightarrow A\left(-\left[\frac{d}{2}\right]-2\right)^{b_2} \oplus A\left(-\left[\frac{d}{2}\right]-3\right)^{b_{v-d-2}} \longrightarrow A\left(-\left[\frac{d}{2}\right]-1\right)^{b_1}$$
$$\oplus A\left(-\left[\frac{d}{2}\right]-2\right)^{b_{v-d-1}} \longrightarrow A \longrightarrow k[\Delta] \longrightarrow 0,$$

where for $1 \leq i \leq v-d-1$,

$$b_i = \binom{v-[\frac{d}{2}]-2}{[\frac{d}{2}]+i}\binom{[\frac{d}{2}]+i-1}{[\frac{d}{2}]}.$$

Even if the geometric realization $|\Delta|$ of a simplicial complex Δ is a sphere, a Betti number of the Stanley-Reisner ring $k[\Delta]$ may depend on the base field k in general. See [T–H$_1$, Example 3.3]. But as for the boundary complexes of cyclic polytopes we have the following result:

COROLLARY 2.3. *Let* Δ *be the boundary complex* $\Delta(C(v,d))$ *of the cyclic polytope* $C(v,d)$, *where* $d \geq 2$. *Then all the Betti numbers of the Stanley–Reisner ring* $k[\Delta]$ *are independent of the base field* k.

We show unimodality of the Betti number sequence $(\beta_0, \beta_1, \cdots, \beta_{v-d})$ of the Stanley-Reisner ring $k[\Delta(C(v,d))]$ associated with $C(v,d)$. Since this sequence is symmetric, i.e., $\beta_i = \beta_{v-d-i}$ for every $0 \leq i \leq v-d$, the unimodality means $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_{[(v-d)/2]}$.

COROLLARY 2.4. *Let* Δ *be the boundary complex* $\Delta(C(v,d))$ *of the cyclic polytope* $C(v,d)$. *Then, the Betti number sequence* $(\beta_0(k[\Delta]), \beta_1(k[\Delta]), \cdots, \beta_{v-d}(k[\Delta]))$ *of the Stanley-Reisner ring* $k[\Delta]$ *over* A *is unimodal.*

§3. Gorenstein complexes

We first give the definition (see [Sta$_1$]). Let k be a field or $\mathbf{Z}$. Let Δ be a $(d-1)$-dimensional simplicial complex with v vertices. We call Δ Gorenstein* over k (or k-Gorenstein*) if it satisfies one of the following equivalent conditions;
(1)For all faces $\sigma \in \Delta$ (including $\sigma = \emptyset$) we have

$$\tilde{H}_i(\text{ link }_\Delta(\sigma);\ k) \cong \begin{cases} k, & \text{if } i = \dim \text{ link }_\Delta(\sigma), \\ 0, & \text{otherwise.} \end{cases}$$

(2)(a)$k[\Delta]$ is Gorenstein.
(b)For all $1 \leq i \leq v$, x_i are zero-divisors in $k[\Delta]$.

We have the following hierarchy;

$$\begin{aligned}&\{\text{Boundary complexes of simplicial polytopes}\}\\ &\subset \{\text{Triangulations of a sphere}\}\\ &\subset \{\mathbf{Z}\text{-Gorenstein* complexes}\}\\ &\subset \{\mathbf{Q}\text{-Gorenstein* complexes}\}.\end{aligned}$$

Remark. (1)All the inclusions above are strict. Non-shellable triangulation of a sphere (a Poincaré sphere, an odd-dimensional real projective space, respectively) gives the first (second, third, respectively) inclusion strictness.
(2)If $v - d \leq 3$, then all the above classes are equal ([Bru–Her$_2$]).
(3)If $d \leq 3$, then all the above classes are equal.

Problem 3.1. Let k be a field. If Δ is a $(d-1)$-dimensional k-Gorenstein* complex with v vertices, then does

$$\beta_i(k[\Delta]) \leq \beta_i(k[\Delta(C(v,d))]).$$

hold? In particular, in the case $d = 3$, does

$$\beta_i(k[\Delta]) \leq \frac{(v-1)(v-i-3)}{i+1}\binom{v-3}{i-1}, \quad 1 \leq i \leq v-4,$$

hold?

Actually it holds if $v - d \leq 3$ ([Bru-Her$_2$]) or if $d \leq 2$. In the case $d = 3$ we can treat the problem from a combinatorial view point because of Remark (3). We have the following partial results;

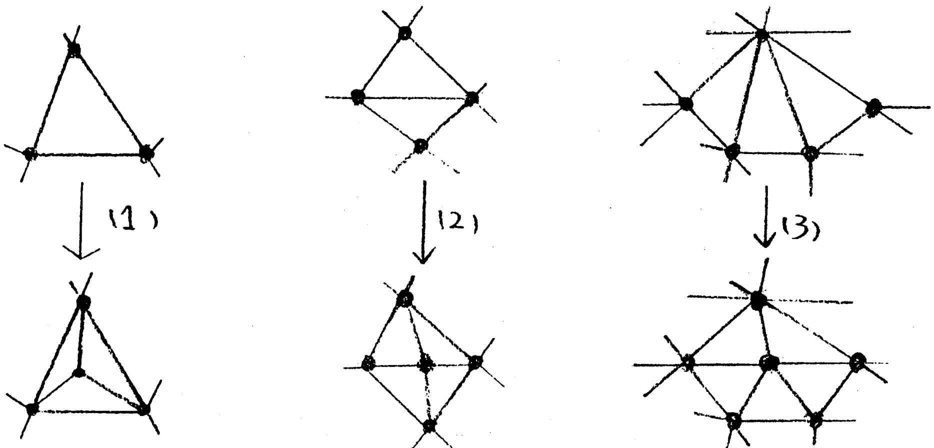
Theorem 3.2. *Let Δ be a 2-dimensional Gorenstein* complex vith v ($\geq$ 5) vertices.*
(1) *We have $\beta_1(k[\Delta]) \leq \frac{(v-1)(v-4)}{2}$.*
(2) *We have*

$$\beta_i \leq \frac{(v-1)(v-i-3)}{i+1}\binom{v-3}{i-1}, \quad 1 \leq i \leq v-4,$$

for $v \leq 12$.

To prove the theorem, we use:

The Induction Theorem of Brücker-Eberhard(cf. [Oda, p190]). *Suppose a finite triangulation Δ of $\mathbf{S}^2$ is given. We get a triangulation Δ' of $\mathbf{S}^2$ with one more vertex, if a vertex of Δ is " split into two " by one of the three steps (1), (2),(3) shown in the figures below. We can obtain any given finite triangulation of $\mathbf{S}^2$ from the tetrahedral triangulation by splitting vertices finitely many times.*



Lemma 3.3. *Let Δ be a triangulation of $\mathbf{S}^2$ with v vertices. And let Δ' be a triangulation obtained from Δ by (1) or (2) in the Induction Theorem above. Then we have*

$$\beta_{i,i+1}(k[\Delta']) \leq \beta_{i,i+1}(k[\Delta]) + \beta_{i-1,i}(k[\Delta]) + \binom{v-3}{i}.$$

for $i \geq 1$.

Lemma 3.4. *Let Δ be a triangulation of $\mathbf{S}^2$ with v vertices. Assume Δ is obtained from the tetrahedral triangulation by successive steps of (1) and (2). Then we have*

$$\beta_{i,i+1}(k[\Delta]) \leq i\binom{v-3}{i+1}.$$

Proof. Thanks to Lemma 3.3, we have

$$\begin{aligned}\beta_{i,i+1}(k[\Delta]) &\leq i\binom{v-4}{i+1} + (i-1)\binom{v-4}{i} + \binom{v-4}{i} \\ &= i(\binom{v-4}{i+1} + \binom{v-4}{i})\end{aligned}$$

$$= i\binom{v-3}{i+1}$$

as required. Q. E. D.

Lemma 3.5. *Let Δ be a triangulation of $\mathbf{S}^2$ with v vertices. If $v \leq 12$, and if Δ is not the icosahedral triangulation, then there exists a vertex x such that $d(x) \leq 4$,* where
$d(x) := \sharp\{\sigma :$ *1-face of* $\Delta \mid x \in \sigma\}$.

Now we skech the proof of Theorem 3.2. First note that non-zero Betti numbers $\beta_{i,j}$ only appear in the 2-linear part $(\beta_{1,2}, \ldots, \beta_{v-4,v-3})$ and in the 3-linear part $(\beta_{1,3}, \ldots, \beta_{v-4,v-2})$ for $1 \leq i \leq v-4$. Since Δ is Gorenstein, we have $\beta_{i,j}(k[\Delta]) = \beta_{v-i-3,v-j}(k[\Delta])$ for every i and j. Put $j := i+2$. We have $\beta_{i,i+2}(k[\Delta]) = \beta_{v-i-3,v-i-2}(k[\Delta])$.

Suppose $v \leq 12$ and Δ is not the icosahedral triangulation, (In the case Δ is the icosahedral triangulation, we must treat it separately. But we omit it here.) By Lemma 3.5, Δ is satisfied the assumption in Lemma 3.4. Then we have

$$\begin{aligned}
\beta_i(k[\Delta]) &= \beta_{i,i+1}(k[\Delta]) + \beta_{i,i+2}(k[\Delta]) \\
&= \beta_{i,i+1}(k[\Delta]) + \beta_{v-i-3,v-i-2}(k[\Delta]) \\
&\leq i\binom{v-3}{i+1} + (v-i-3)\binom{v-3}{v-i-2} \\
&= \frac{(v-1)(v-i-3)}{i+1}\binom{v-3}{i-1},
\end{aligned}$$

which is the assertion (2).

As for the assertion (1), note that $\beta_{1,2}(k[\Delta]) = \frac{(v-3)(v-4)}{2}$, which only depends on v. Hence we have only to show $\beta_{1,3}(k[\Delta]) \leq \frac{v-4}{2}$. Then we have only to check that the number of "empty triangles (circles of length 3 which are not 2-faces)" increases at most by one in the case (3) in the Induction Theorem, which is immeadiate.

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